

# Variational Principles for Constrained Electromagnetic Field and Papapetrou Equation

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## Abstract

In our previous article [4] an approach to derive Papapetrou equations for constrained electromagnetic field was demonstrated by use of field variational principles. The aim of current work is to present more universal technique of deduction of the equations which could be applied to another types of non-scalar fields. It is based on Noether theorem formulated in terms of Cartan' formalism of orthonormal frames. Under infinitesimal coordinate transformation the one leads to equation which includes volume force of spin-gravitational interaction. Papapetrou equation for vector of propagation of the wave is derived on base of the equation. Such manner of deduction allows to formulate more accurately the constraints and clarify equations for the potential and for spin.

## 1 Introduction

Complete description of motion of a non scalar wave in gravitational field is given by its covariant field equations. Under quasi-classical consideration when wave length is sufficiently less than typical scales of observations, propagation of the wave is substituted by motion of a particle. A. Papapetrou in early 50s showed that spin-gravitational interaction changes form of the trajectory of a particle with spin. Deduction equations of motion of the quantum particle with account of spin-gravitational interaction can be provided by construction its classical relativistic Lagrangian. This problem

has chained a great interest of researchers for last decades. Nevertheless a satisfactory description was not obtained, see [1, 3].

Evident progress in theoretical treatment of the problem for photons was achieved in series of our recent articles [2, 3, 4, 5]. Particularly in the work [4] an approach to deduction Papapetrou equation for photon based on field variational principle was demonstrated. Essence of that approach was attachment the field of potential to congruence of 0-curves which specify propagation of the wave. The congruence and the field were described by frame  $\{\vec{n}_\pm, \vec{n}_{1,2}\}$  which is orthonormal in sense that  $\langle \vec{n}_+, \vec{n}_- \rangle = 1$ ,  $\langle \vec{n}_\alpha, \vec{n}_\alpha \rangle = -1$ ,  $\alpha = 1, 2$  are only non zero scalar products of the vectors. Vector  $\vec{n}_-$  is tangent to the 0-curves everywhere and vectors  $\vec{n}_{1,2}$  constitute polarization tangent subspace [4]. Integrand of the action was presented as quadric form over derivatives of the potential. Under the constraints imposed the one was reduced to form similar to action of classic particle with an additional term describing spin-gravitational interaction. Next it was shown that varying this action under infinitesimal dragging changing shape of the 0-curves leads to Papapetrou equation for shape of the 0-curves. Variation components of the potential attached to the curves reduced equation for potential to  $D_- A^\alpha = 0$ . That in turn led to equation for spin:  $D_- S_{\alpha\beta} = 0$ , see [4].

In this article we are returning to consideration of the problem pursuing the aim to create more formal and universal approach to deduct quasi-classical equations for motion of photon under account of spin-gravitational interaction. At the beginning we explain technic of variation in framework of theory of exterior differential forms. Next we develop and improve method demonstrated in paper [4]. In particular, conception that elements of the orthonormal frame are variables of Lagrangian describing congruence of curves of propagation of the wave fruited one of main ideas of the current work. That is to consider elements of the frame as variables of gravitational field. Then to apply infinitesimal coordinate transformations which transform orthonormal frames, connection coefficients, potential of electromagnetic field but leave invariant gravitational field. As usual Noether theorem gives differential equations for the field observables like stress-energy tensor (SET) and current of spin. However, this time the equations do not exhibit covariant conservation law for the SET, but express equation of dynamic of electromagnetic wave. Left hand side (LHS) of the one is usual covariant divergence of the SET, but in the right hand side (RHS) instead zero a term interpreted as volume force of spin gravitational interaction arises. It happens due to

vector character of electromagnetic field.

Then we develop our quasi-classical approach. It is based on assumption of existence a class of locally monochromatic (LM) electromagnetic waves which admit geometric optical description. It means that such the wave propagates along congruency of worldlines satisfying an equation whose LHS coincides with equation of geodesic line in a canonical comoving frame [4]. The equation should be derived from the equation of dynamic. Conditions which reduce the one to equation for shape of worldlines specify the class of LM electromagnetic waves. Analysis LHS of the equation of dynamic gives conditions as follows.  $T_{++}$  should be only non vanishing component of SET of the wave and equation of continuity for probability current had to be satisfied. In turn RHS of the equation of dynamic should include characteristics of the particle but not the ones of the field. It suggests that current of spin can be expressed via tensor of spin and vector of propagation:  $S_{ab(c)} = \delta_c^+ S_{ab}$ , besides  $S_{12}$  only non zero component of the tensor of spin due to definite helicity of photon. By other words we demand that LM electromagnetic wave should be transversal wave. Such the requirements are provided by transversality of the potential and choice of suitable form of the Lagrangian.

Analysis of form of the SET and restrictions put onto it gives constraints which should be imposed to the potential of the field. The constraints coincide with the ones were imposed in [4]. Condition of continuity of probability current finds its application under deduction of equation for components of the potential. Now  $D_- A^\alpha \neq 0$  but proportional to  $A^\alpha$ ,  $\alpha = 1, 2$  and shows attenuation of the wave caused by divergence of the beam of electromagnetic wave. Nevertheless it does not change our main conclusions obtained earlier [4, 5]. Specifically, under redefinition the spin as normalized  ${}^*\nu^+$  component of current of spin equation for spin leaves the same.

## 2 Lagrange formalism in framework of orthonormal frame

Action of electromagnetic field in curved space-time usually expressed by integral of Lagrangian density  $\mathcal{L}$  multiplied to 4-form  $\epsilon$  of unit volume. For aims of our studies it is convenient to express the integrand as whole 4-form  $\Lambda$  which we call 4-form of Lagrangian:

$$\mathcal{A} = \int \mathcal{L} \epsilon = \int \Lambda, \quad \epsilon = \sqrt{-g} dx^0 \wedge \dots \wedge dx^3 = \nu^0 \wedge \dots \wedge \nu^3, \quad (1)$$

where  $\{x^i\}$  some coordinates in considered domain of space-time and  $\{\nu^a\}$  is a field of orthonormal frames dual to  $\{\vec{n}_b\}$ . In general the 4-form of Lagrangian:

$$\Lambda = \Lambda(A_b, DA_c, \nu^a), \quad (2)$$

depend on components of field potential  $A_b(x^i)$ , their derivatives and elements of orthonormal frame  $\{\nu^b\}$  which describes gravitational field. Besides, it is convenient to represent derivatives of the potential via covariant exterior derivatives (CED):

$$DA_b = (D_a A_b)\nu^a = dA_b - \omega_b{}^c A_c, \quad (3)$$

where symbol  $d$  stands for exterior derivative and  $\omega_b{}^c$  is 1-form of Cartan' rotation coefficients. Notion of CED and rules of operating with it should be briefly observed. We define CED of any exterior differential form with (exterior) tensor indexes as follows:

$$\begin{aligned} D(T_{\dots a \dots}{}^{\dots b \dots}) &= dT_{\dots a \dots}{}^{\dots b \dots} + \dots \\ &- \omega_a{}^c \wedge T_{\dots c \dots}{}^{\dots b \dots} + \dots - \omega_{\dots}{}^b \wedge T_{\dots a \dots}{}^{\dots c \dots} + \dots \end{aligned}$$

Usual Leibnitz rules can be easily generalized for the CED under account of antisymmetry of exterior product of the differential forms. We also define CED of vector:

$$D\vec{n}_a = \vec{n}_b \otimes \omega_a{}^b; \quad D\vec{\varepsilon} = \vec{n}_a \otimes D_b(\varepsilon^a)\nu^b,$$

where symbol  $\otimes$  stands for tensor product of elements of tangent and cotangent spaces. Thanks to definitions it is seen that rules as follows are valid:

$$\vec{n}_a \otimes D(T^a) = D(\vec{n}_a) \wedge T^a + \vec{n}_a \otimes dT^a. \quad (4)$$

In terms of 4-form of the Lagrangian the field equations have a form as follows:

$$\frac{\partial \Lambda}{\partial A_b} - D \left( \frac{\partial \Lambda}{\partial DA_b} \right) = 0, \quad (5)$$

where  $\partial \Lambda / \partial A_b$  is a partial derivative of 4-form of the Lagrangian over field variable  $A_b(x)$ . An algebraic derivative  $\partial \Lambda / \partial DA_b$  of 4-form  $\Lambda$  over 1-form  $DA_b$  [6] reduces it to 3-form. CED of the algebraic derivative restores degree of the differential form up to 4th. Notion of algebraic derivative provides

compact view of formulae especially under consideration variations of the 4-form of Lagrangian like follows:

$$\begin{aligned}\Lambda(\dots, DA + \delta DA, \dots) - \Lambda(\dots, DA, \dots) &= \delta_{DA}\Lambda = \delta DA \wedge \frac{\partial \Lambda}{\partial DA}, \\ \delta_\nu \Lambda &= T_{ab} \langle \delta \nu^a, \nu^b \rangle \epsilon = T_{ab} \delta \nu^a \wedge * \nu^b = \delta \nu^a \wedge \frac{\partial \Lambda}{\partial \nu^a};\end{aligned}\quad (6)$$

where  $\delta DA$  is variation of CED of potential of the electromagnetic field,  $\delta \nu^a$  expresses variation of gravitational field variables,  $T_{ab}$  is stress-energy tensor (SET) of electromagnetic field, asterisk stands for Hodge conjugation of exterior differential forms.

### 3 Noether theorem and Spin current

Under infinitesimal rotations of field of the orthonormal frames  $\{\nu^b\}$ :

$$\delta \nu^a = -\varepsilon_b{}^a \nu^b, \quad \varepsilon_{ab} + \varepsilon_{ba} = 0, \quad (7)$$

variation of CED of the potential becomes:

$$\delta DA_c = D\delta A_c + \delta \omega_{ab} \frac{\partial DA_c}{\partial \omega_{ab}}. \quad (8)$$

Due to field equations, only part of variation of the Lagrangian contributing to variation of the action is:

$$\delta \omega_{ab} \frac{\partial DA_c}{\partial \omega_{ab}} \wedge \frac{\partial \Lambda}{\partial DA_c} = \frac{1}{2} \delta \omega_{ab} \frac{\partial DA_c}{\partial \omega_{[ab]}} \wedge \frac{\partial \Lambda}{\partial DA_c} = \frac{1}{2} \delta \omega_{ab} \wedge S_{\dots c}^{ab} * \nu^c, \quad (9)$$

where square brackets means antisymmetrization and  $S_{\dots c}^{ab} * \nu^c$  stands for current of spin of the field. Variation of the connection 1-form under the considered rotations is:

$$\delta \omega_{ab} = -D\varepsilon_{ab},$$

where  $D\varepsilon_{ab} = d\varepsilon_{ab} - \omega_a{}^c \varepsilon_{cb} - \omega_b{}^c \varepsilon_{ac}$  is CED of tensor of rotations. Substituting it into (9) represents variation of (1) as follows:

$$\delta \mathcal{A} = \int d[\dots] + \int \frac{1}{2} \varepsilon_{ab} D[S_{\dots c}^{ab} * \nu^c] = 0.$$

This gives us covariant equation of continuity of 3-form of current of spin:

$$D[S_{\dots c}^{ab} * \nu^c] = 0. \quad (10)$$

## 4 Coordinate variation

Infinitesimal vector field  $\vec{\varepsilon}$  drags coordinate hyper-surfaces  $\{x^i\}$  onto  $\{y^i\}$  and elements of orthonormal covector and vector frames  $\{\vec{n}_a\}$ ,  $\{\nu^b\}$  onto dragged frames  $\{\vec{n}_a\}$ ,  $\{\nu^b\}$ . Applying this variation to action of Gravitational field gives equation of covariant continuity for Einstein tensor in natural vector frame and in orthonormal one:

$$G_{i\cdot|j}^{\cdot j} = D_b(G_a^{\cdot b}) = 0.$$

The same procedure can be performed for (1) as follows:

$$\int \delta \Lambda = \int \Lambda(\vec{n}_0, \vec{n}_1, \vec{n}_2, \vec{n}_3) \nu^0 \wedge \dots \nu^3 - \int \Lambda(\vec{n}_0, \dots) \nu^0 \wedge \dots \nu^3, \quad (11)$$

where  $\Lambda(\vec{n}_0, \dots)$  means value of 4-form  $\Lambda$  on 4-vector  $[\vec{n}_0 \vec{n}_1 \vec{n}_2 \vec{n}_3]$ . Under this definition variations of variables of the Lagrangian are given by Lie derivatives over vector field  $\vec{\varepsilon}$ . Use of Cartan' first and second structure equations expresses variations of  $\nu^a$  and  $\omega_a^{\cdot b}$  as follows:

$$\delta \nu^a = d\nu^a(\vec{\varepsilon}) + d\varepsilon^a = D\varepsilon^a - \omega_b^{\cdot a}(\vec{\varepsilon})\nu^b, \quad (12)$$

$$\delta \omega_a^{\cdot b} = d\omega_a^{\cdot b}(\vec{\varepsilon}) + d[\omega_a^{\cdot b}(\vec{\varepsilon})] \pm [\omega_c^{\cdot b} \wedge \omega_a^{\cdot c}](\vec{\varepsilon}) = \Omega_a^{\cdot b}(\vec{\varepsilon}) + D[\omega_a^{\cdot b}(\vec{\varepsilon})]; \quad (13)$$

where  $\Omega_a^{\cdot b} = 1/2 R_{a\cdot cd}^{\cdot b} \nu^c \wedge \nu^d$  is 2-form of curvature. This way variation of 4-form of Lagrangian (2) becomes:

$$\delta_A \Lambda + \delta_\nu \Lambda + \delta_{DA} \Lambda = \delta A_b \frac{\partial \Lambda}{\partial A_b} + \delta D A_b \wedge \frac{\partial \Lambda}{\partial D A_b} + \delta \nu^a \wedge \frac{\partial \Lambda}{\partial \nu^a}$$

Noting that variation of CED as before is given by (8) where this time  $\delta \omega$  is given by (13) we after applying (5) and (9) rewrite the above expression as follows:

$$\frac{1}{2} \{ \Omega_{ab}(\vec{\varepsilon}) + D[\omega_{ab}(\vec{\varepsilon})] \} \wedge S^{\cdot ab}_{\cdot c} \nu^c + D(\varepsilon^a) \wedge T_{ab} \nu^b, \quad (14)$$

where only first term of variation of  $\nu^a$  (12) contributes owing to symmetry of the SET. Substituting the above expression into (11) we integrate by parts. Referring to (10) we obtain:

$$\int d[\dots] + \int \{ 1/2 \Omega_{ab}(\vec{\varepsilon}) \wedge S^{\cdot ab}_{\cdot c} \nu^c - \varepsilon^a D [T_{ab} \nu^b] \} = 0.$$

It brings us equation as follows:

$$\vec{n}_c \otimes D \left[ T^c_{\cdot b} {}^* \nu^b \right] = 1/2 \vec{n}_c \otimes R_{ab\cdot\cdot}{}^{cd} S^{\cdot\cdot ab}_{\cdot\cdot d} \epsilon. \quad (15)$$

The one in contrast with analogous equations for scalar fields is not equation of covariant conservation of the SET due to non zero RHS appeared. The RHS expresses volume force of spin-gravitational interaction while the LHS exhibits CED of current of stress-energy-momentum. Because of this we will call (15) as equation of dynamic of the electromagnetic wave in curved space-time. Accordingly to rule (4) LHS of (15) can be represented as:

$$\vec{n}_c \otimes D \left[ T^c_{\cdot b} {}^* \nu^b \right] = D(\vec{n}_c) \wedge T^c_{\cdot b} {}^* \nu^b + \vec{n}_c \otimes d \left[ T^c_{\cdot b} {}^* \nu^b \right]. \quad (16)$$

It should be marked that use of the above mentioned rule becomes possible owing to fact that all indexes in the above expression are dummy. Such form of presentation of covariant divergence of the SET is convenient for transition to quasi classical consideration.

## 5 Quasi classical approach, Schweber Lagrangian and constraints

Conception of our quasi classical description is a deduction of equation for vector of propagation  $\vec{n}_-$  from the equation of dynamic (15). Besides we demand that obtained equation becomes coincide with equation of geodesic line under switch off the spin. This means that covariant divergence of SET should be reduced to covariant derivative of  $\vec{n}_-$ . Equality (16) shows that this result is provided by constraints as follows:

$$T_{++} \text{ is only non vanishing component of the SET}, \quad (17)$$

$$d \left[ T_{++} {}^* \nu^+ \right] = 0. \quad (18)$$

It should be distinguished that (17,18) characterize subspace of locally monochromatic (LM) electromagnetic waves and class of canonical comoving frames [4] which allows most convenient form of quasi-classical equations for the waves. The same form of SET has monochromatic electromagnetic wave in flat space-time propagating along  $\vec{n}_-$ :

$$T^{ab} = \delta_-^a \delta_-^b \omega^2 |\vec{A}|^2,$$

where  $\omega$  is frequency of the wave. Thus (18) expresses continuity of current of probability in case of flat space-time. Under this conditions LHS of (15) becomes:

$$D[\vec{n}_-] \wedge T_{++}^* \nu^+ = T_{++}^- D_-[\vec{n}_-] \epsilon.$$

This way we reduce (15) to sought form:

$$T_{++} D_-[\vec{n}_-] = \vec{n}_c R_d{}^c{}_{ab} S^{abd}. \quad (19)$$

LHS of the equation coincides with geodesic equation for vector field  $\vec{n}_-$ , however in the RHS stands a term including contraction of the curvature with current of spin of the field which we call force of spin-gravitational interaction. Besides the force still is written in terms of field theory. To pass to complete quasi classical description we need to determine field Lagrangian.

Under study process of propagation of LM electromagnetic wave it is relevant admit Schweber Lagrangian [7] which provides adequate quasi-classical description of LM wave:

$$\mathcal{L} = 1/2 D_a A^c D_b A_c < \nu^a, \nu^b >. \quad (20)$$

In terms of orthonormal frames we write 4-form of the Lagrangian as follows:

$$\Lambda = 1/2 D A_c(\vec{n}_a) \{ D A^c \wedge * \nu^a \}. \quad (21)$$

Now let's calculate variation of the action under variation of elements of the frame:

$$\delta_\nu \Lambda = 1/2 D A_c(\delta \vec{n}_a) D A^c \wedge * \nu^a + 1/2 D A_c(\vec{n}_a) D A^c \wedge \delta * \nu^a. \quad (22)$$

It is evident that:  $\delta \vec{n}_a = -\delta \nu^b(\vec{n}_a) \vec{n}_b$ . So first term in (22) can be rewritten as follows:

$$-1/2 < \delta \nu^b, \nu^d > D_b A_c D_d A^c.$$

During variation of the second term asterisk conjugation should be varied:

$$\begin{aligned} \delta * \nu^a &= 1/3! \varepsilon^a{}_{bcd} \delta(\nu^b \wedge \nu^c \wedge \nu^d) = 1/2 \varepsilon^a{}_{bcd} \delta \nu^b \wedge \nu^c \wedge \nu^d \\ \Rightarrow \nu^e \wedge \delta * \nu^a &= -1/2 \varepsilon^a{}_{bcd} \varepsilon^{ecd}{}_{\dots f} \delta \nu^b \wedge * \nu^f = (\eta^{ae} \eta_{bf} - \delta_f^a \delta_b^e) < \delta \nu^b, \nu^f > \epsilon. \end{aligned}$$

This brings us:

$$\delta_\nu \Lambda = - \left( D_a A_c D_b A^c - 1/2 D_e A_f D^e A^f \eta_{ab} \right) < \delta \nu^a, \nu^b > \epsilon$$



By other words, accordingly to (6), we find:

$$T_{ab} = - \left( D_a A_c D_b A^c - 1/2 D_e A_f D^e A^f \eta_{ab} \right). \quad (23)$$

The next part of the variation contributing to RHS of (15) is variation over Cartan' rotation 1-form. The one appears in expression of CED of potential (3). Evidently  $\delta_\omega D A_c = -\delta\omega_c{}^d A_d$ , so

$$\delta_\omega \Lambda = \delta_\omega D A_c \wedge \frac{\partial \Lambda}{\partial D A_c} = \delta\omega_c{}^d \wedge A_d D_b A^c {}^* \nu^b = 1/2 \delta\omega^{cd} \wedge A_{[d} D_b A_{c]} {}^* \nu^b.$$

Now equating this expression to (9) we obtain:

$$S_{cdb} = A_{[d} D_b A_{c]}. \quad (24)$$

Since photons has definite helicity they are presented by electromagnetic waves with circular polarization [4]. Hence it is convenient to consider complex valued amplitudes of the potential which describe phase shift of circularly polarized waves. For this aim an ordinary procedure of redefinition the observables being quadric forms of the potential components is used. That is  $A_p A_q \rightarrow 1/2 \bar{A}_{\{p} A_{q\}}$ , where curly brackets mean symmetrization over the indexes. After applying it to expressions for the SET and the current of spin we obtain their form in complex amplitudes:

$$T_{ab} = -\frac{1}{2} \left\{ D_a \bar{A}_c D_b A^c + D_a A_c D_b \bar{A}^c - \eta_{ab} D_c \bar{A}_d D^c A^d \right\}, \quad (25)$$

$$S_{cdb} = \frac{1}{2} \left( \bar{A}_{[d} D_b A_{c]} + A_{[d} D_b \bar{A}_{c]} \right). \quad (26)$$

Now let us find constraints should be put to the potential. As we consider LM electromagnetic wave we expect that potential of the wave has a structure as follows:

$$\vec{A} = \vec{a} e^{i\phi}, \quad D_+ \phi = \omega, \quad D_- \phi = 0, \quad D_+ \vec{a} = 0 \quad (27)$$

$$\Rightarrow D_+ A_\alpha = i\omega A_\alpha. \quad (28)$$

Due to (25) it is seen that condition of dominance of  $T_{++}$  demands vanishing derivatives of the potential along polarization vectors:

$$D_\alpha \vec{A} = 0, \quad \alpha = 1, 2. \quad (29)$$

Vanishing the derivative along direction of propagation  $\vec{n}_-$  makes the condition surely satisfied, however it is not only possibility as it will be shown later. Anyhow value of  $T_{++}$  can be calculated now due to fact that  $\eta_{++} = 0$ . Under constraints imposed it is reduced to the form of  $T_{++}$  of monochromatic wave in flat space-time:

$$T_{++} = -\omega^2 < \vec{A}, \vec{A} > = \omega^2 |\vec{A}|^2 \quad (30)$$

Quasi classical picture assumes that spin of photon moves along field of vectors  $\vec{n}_-$  and has only  $S_{12}$  component due to fact of definite helicity of photon. Constraints (29,28) provide validity of the first condition which can be written as  $S_{cdb} = S_{cd} \eta_{b-}$ , where  $S_{cd}$  is tensor of spin. The second condition is supplied by additional constraints:

$$A_{\pm} = 0. \quad (31)$$

## 6 Field equations

Under varying Lagrangian (20) over components of potential we obtain reduced form of Maxwell equations:

$$D \left[ D_b A^c {}^* \nu^b \right] = 0.$$

Let's consider equations for nonzero components of the potential. After imposing constraints we obtain:

$$D_{\{+} \left( D_{-} \} A^{\alpha} \right) \epsilon + \sum_{c=\pm} D_c A^{\alpha} d^* \nu^c = 0. \quad (32)$$

As usual under quasi classical (geometric optical) consideration we assume that values of second derivatives of amplitudes of the potential should be neglected:

$$\omega^{-1} |D_a (D_b a^c)| = \omega^{-1} O \left( |\vec{a}|^{-3} \left[ D_- |\vec{a}|^2 \right]^2 \right) = o \left( |\vec{a}|^{-1} D_- |\vec{a}|^2 \right).$$

It reduces (32) to:

$$[2i\omega D_- + i(D_- \omega)] A^{\alpha} \epsilon + i\omega A^{\alpha} d^* \nu^+ + (D_- A^{\alpha}) d^* \nu^- = 0. \quad (33)$$

Complex conjugation of (33) gives equation for complex amplitude. Let's consider contraction of (33) with  $\bar{A}_\alpha$  and contraction of the one complex conjugated with  $A_\alpha$ . Difference between them gives:

$$i \left[ \omega D_- \langle \vec{A}, \vec{A} \rangle + (D_- \omega) \langle \vec{A}, \vec{A} \rangle \right] \epsilon + i\omega \langle \vec{A}, \vec{A} \rangle d^* \nu^+ + \\ + 1/2 \left[ \bar{A}_\alpha D_- A^\alpha - A_\alpha D_- \bar{A}^\alpha \right] d^* \nu^- = 0.$$

Due to structure of potential and constraints we daresay that last term in the above equation is zero. This manner we rewrite the equation as follows:

$$\omega^{-1} d[\omega^2 \langle \vec{A}, \vec{A} \rangle^* \nu^+] - (D_- \omega) \langle \vec{A}, \vec{A} \rangle = 0.$$

But owing to (30) and (18) first term in the equation is zero. It gives  $D_- \omega = 0$ . By other words value of derivative of the frequency in chosen frame along vector  $\vec{n}_-$  is vanishing in quasi classical approximation:

$$D_- \omega = O \left( \left[ \frac{D_- \langle \vec{a}, \vec{a} \rangle}{\langle \vec{a}, \vec{a} \rangle} \right]^2 \right) = o \left( \frac{\omega D_- \langle \vec{a}, \vec{a} \rangle}{\langle \vec{a}, \vec{a} \rangle} \right). \quad (34)$$

This result allows to rewrite (33) and (18) as follows:

$$i\omega \left[ 2D_- A^\alpha \epsilon + A^\alpha d^* \nu^+ \right] + (D_- A^\alpha)^* d^* \nu^- = 0, \quad (35)$$

$$d \left[ \langle \vec{A}, \vec{A} \rangle^* \nu^+ \right] = 0. \quad (36)$$

Equating to zero factor at  $\omega$  in (35) as it is used to do under geometric optical approximation we obtain equation for potential components. In turn (36) expresses conservation of probability current of the wave and allows to exclude  $d^* \nu^+$  from the equation for the potential:

$$D_- A^\alpha = 1/2 A^\alpha d^* \nu^+ = A^\alpha \frac{D_- (\bar{A}_c A^c)}{2\bar{A}_c A^c}. \quad (37)$$

Equation (37) describes attenuation of the amplitude of potential caused by divergence of worldlines of propagation of the wave.

## 7 Explicit form of SET, current of spin and Papapetrou equation

Substituting (37) together with constraints into (25,26) allows us to obtain an explicit form of elements of SET:

$$D_a \bar{A}_b D^a A^b = D_{\{+} \bar{A}_b D_{- \} A^b = [-i\omega \bar{A}_b A^b / 2 + 1/2 \bar{A}_b i\omega A^b] \frac{D_- \bar{A}_c A^c}{\bar{A}_c A^c} = 0,$$

$$T_{+-} = -1/2 D_{\{+} \bar{A}_c D_{- \} A^c = 0, \quad T_{--} = -D_- \bar{A}_b D_- A^b \ll T_{++}.$$

The calculations confirm assumptions about view of elements of SET of LM wave. Next step is to calculate elements of spin tensor. Thanks to constraints all its elements with third index different from "±" is surely zero. But accordingly to (26):

$$\begin{aligned} S_{cd+} &= i\omega \bar{A}_{[d} A_{c]} =: S_{cd}, \\ S_{cd-} &= \frac{D_- |\vec{A}|^2}{|\vec{A}|^2} (\bar{A}_{[d} A_{c]} + A_{[d} \bar{A}_{c]}) = 0, \\ \Rightarrow S_{cdb} &= \delta_b^+ S_{cd}. \end{aligned} \tag{38}$$

It approves expected factorized form of current of spin. Now substituting (30) and (38) into (19) we obtain equation for vector  $\vec{n}_-$  being tangent to worldlines of propagation of the wave:

$$\omega^2 |\vec{A}|^2 D_- \vec{n}_- = 1/2 \vec{n}_c R^{abc+} S_{ab}. \tag{39}$$

In fact the one coincides with Papapetrou equation for trajectory of photon obtained us in work [4], although combinatorial factor 1/2 was not present there. It happened only by a variety in the definitions of tensor  $S_{ab}$  in the articles. Under practical calculations it is more convenient to introduce normalized tensor of spin [5]:

$$\sigma_{ab} = \frac{S_{ab}}{\omega |\vec{A}|^2} = \frac{\bar{A}_{[a} A_{b]}}{i |\vec{A}|^2}. \tag{40}$$

As expected, the one is transporting parallel itself in polarization tangent subspace:

$$\begin{aligned} D_- \sigma_{\alpha\beta} &= i \left[ \frac{D_- (\bar{A}_{[\alpha} A_{\beta]}) + \bar{A}_{[\alpha} D_- (A_{\beta]})}{\bar{A}_c A^c} - \frac{\bar{A}_{[\alpha} A_{\beta]} D_- (\bar{A}_c A^c)}{(\bar{A}_d A^d)^2} \right] = \\ &= i \left[ \frac{D_- (\bar{A}_c A^c)}{2(\bar{A}_d A^d)^2} \{ \bar{A}_{[\alpha} A_{\beta]} + \bar{A}_{[\alpha} A_{\beta]} \} - \frac{\bar{A}_{[\alpha} A_{\beta]} D_- (\bar{A}_c A^c)}{(\bar{A}_d A^d)^2} \right] = 0. \end{aligned}$$

Finally excluding potentials of the electromagnetic field, we write system of Papapetrou equations for vector of propagation of LM electromagnetic wave and its spin as follows:

$$D_- \sigma^{\alpha\beta} = 0, \quad \omega D_- (\vec{n}_-) = 1/2 \vec{n}_c R_{\alpha\beta}^c \sigma^{\alpha\beta}; \quad (41)$$

where value of frequency  $\omega$  leaves constant in canonical comoving frame along the trajectory with accuracy up to second order in accordance with [4].

## 8 Remarks

It should be underlined that idea of this work is grounded on conception of article [4]. Specifically, in the above cited paper we started by consideration gauge invariant Lagrangian  $\mathcal{L} = 1/2 D_a A_b D_c A_d \langle \nu^a \wedge \nu^b, \nu^c \wedge \nu^d \rangle$ . Next we expand scalar products as  $\langle \nu^a, \nu^{[c} \rangle \langle \nu^b, \nu^{d]} \rangle$  and put the constraints. It was guessed that only terms  $\langle \nu^a, \nu^c \rangle \langle \nu^b, \nu^d \rangle$  would contribute to variation of the action due to fact that they yields physically interpretable expressions. In present work that assumption finds its expression in choice of form of Lagrangian. From another point of view selection of Schweber Lagrangian together with the constraints serves transversality of LM electromagnetic wave.

We also note that variation of omitted scalar product in Schweber Lagrangian  $D_a A_c D_b A^c \rightarrow D_a A_c D_b A_d \langle \nu^c, \nu^d \rangle$  may contribute only to the SET. Calculations with account of the constraints and symmetrization over complex amplitudes analogous to the ones performed in section 7 shows that the contribution vanishes.

Equation for probability current (36) can be rewritten as:

$$D_- \langle \vec{A}, \vec{A} \rangle + \langle \vec{A}, \vec{A} \rangle Div \vec{n}_- = 0,$$

where  $Div$  stands for covariant divergence of the vector. Under our approximation we daresay that both terms in the equation are small but not vanishing. But in our former work [4] it was implicitly assumed that (36) is satisfied by vanishing of both of its terms. Hence equation for amplitudes of the potential obtained us earlier is particular case of (37). The last is realized when worldlines of propagation of the wave are parallel themselves locally, as it was demanded in the previous article.

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## References

- [1] A. Frydryszak, *Lagrangian Models of Particles with Spin: the First Seventy Years*/ arXiv:hep-th/9601020
- [2] Turakulov Z Ya, Safonova M, *Motion of a Vector Particle in a Curved Space-Time. I. Lagrangian Approach*/ Mod Phys Lett A18 (2003) 579
- [3] Turakulov Z Ya, Safonova M, *Motion of a Vector Particle in a Curved Space-Time. II. First-Order Correction to a Geodesic in a Schwarzschild background*/ Mod Phys Lett A 20 (2005) 2785
- [4] Turakulov Z. Ya, Muminov A. T, *Electromagnetic field with constraints and Papapetrou equation*/ Zeitschrift fur Naturforschung 61a, 146 (2006)
- [5] Turakulov Z Ya, Muminov A T, *Motion of a Vector Particle in a Curved Space-Time. III. Development of Techniques of Calculations*/ Mod Phys Lett A21 n 26 (2006) 1981
- [6] Bazylev V.T, *Geometry of Differentiable Manifolds*. Vysshaya Shkola, Moscow 1989
- [7] Mitskievich N.V, *Physical Fields in General Relativity Theory*. Nauka, Moscow (1969)